Ex 11.

1. b) Yes. Denote all $n \times n$ matrix by $M_{n\times n}$.

$\pi_0(M_{n\times n}) = 1$ (as $M_{n\times n} \overset{\text{home}}{\cong} \mathbb{R}^{n^2}$)

(i.e ~~conn~~ $M_{n\times n}$ conn)

c) No., as $GL(n;\mathbb{R}) = \{ M \in M_{n\times n} \mid \det M \neq 0 \}$.

$GL(n;\mathbb{R}) \xrightarrow{\det} \mathbb{R}$.

$\shortparallel$

Not conn as $\operatorname{Im}(\det) = \mathbb{R} \setminus \{0\}$.

d) Yes. $H(x,t) = (1-t) f(x) + t g(x)$

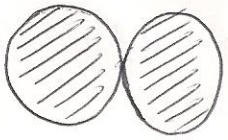
4. $f_t(z) = z e^{2\pi i t |z|}$ $\quad t \in [0,1]$

5. $f_t(z) = z e^{2\pi i t(|z|-1)}$

Ex 10.

2. Connectedness of $(X, \mathcal{T}_2)$ implies conn of $(X, \mathcal{T}_1)$

Conversely, e.g $\mathcal{T}_{std} \subseteq \mathcal{T}_{discrete}$ on $\mathbb{R}$

3. b) Not true 

c) If A conn, $Frt(A)$ may not be conn.

Eg: $X = \mathbb{R}^2$, $A = \mathbb{R}^2 \setminus \{(0,0), (0,1)\}$, $Frt(A) = \{(0,0), (0,1)\}$

If $Frt(A)$ conn, A may not be conn

Eg. $X = \mathbb{R}$. $A = \mathbb{R} \setminus \{0\}$ $Frt(A) = \{0\}$

6. Eg1: $(X, \mathcal{T}_{discrete})$ Eg2: $\{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$ w/ induced topo from $\mathbb{R}$

8. The converse is Not true

Eg. $\mathcal{T}_{\mathbb{R}}$: Standard topo on $\mathbb{R}$. $\mathcal{T}_{\{0,1\}} = \{\phi, \{0,1\}\}$

$f: \mathbb{R} \to \{0,1\} \times \mathbb{R}$. $f(x) = (0, x)$

$g: \mathbb{R} \to \{0,1\} \times \mathbb{R}$. $g(x) = (1, x)$

Then $G_f \cap G_g = \phi$. $G_f \cup G_g$ is conn. ($\{0,1\} \times \mathbb{R}$ is endowed w/ product topo)

7. If G_f is conn, it is true that X is conn

10. b) No. $f: \mathbb{R} \to \mathbb{R}$

$x \mapsto x^2$.

Ex 9 (b)

1. It is obvious that proper subset of an infinite set (w/ discrete topo) may not be compact

2. Eg: ~~$(X = \mathbb{R}, \mathcal{T}_{co\text{-}countable})$ it is non-T_2~~

~~consider $x = 1$ & $U = \mathbb{R} \setminus \{2\}$~~

$= [0,1] \times \{p, q\}$ where $t \times p \sim t \times q$ for $\forall\, t \neq \frac{1}{2}$

4. Eg 1: $(\mathbb{R}, \mathcal{T}_{discrete}) \xrightarrow{Id} (\mathbb{R}, \mathcal{T}_{trivial})$ ← Non-T_2.

$\mathcal{T}_{trivial} = \{\phi, \mathbb{R}\}$

Eg 2: $([0,1], \mathcal{T}) \xrightarrow{Id} ([0,1], \mathcal{T}_{std})$

where $\mathcal{T} \supsetneq \mathcal{T}_{std}$, then $([0,1], \mathcal{T})$ is Non-cpt. !

5. If the set is finite, it is T_2

6. a) Y is T_1

b) Y is T_2.

Eg: consider $\mathbb{Q}$ with induced topo from $(\mathbb{R}, \mathcal{T}_{std})$
it is T_2 & any cpt subset have empty interior

Denote $\mathbb{Q}^*$ to be its one-pt compactification
Note that "the one-pt cpt^n of X" is T_2 iff X is T_2 & locally cpt.

Ref: Wiki "Locally cpt space"
"Alexandroff extension"

Ex 9 (a)

1. c) No for co-countable topo

5. $X = \mathbb{N}$, $\mathcal{T}_X = \{\phi, \mathbb{N}\}$

$Y = \{2n \mid n \in \mathbb{N}\}$ is compact but Not closed

7. Converse is Not true. Eg: $X = (0,1)$, $\mathcal{T}_X = \mathcal{T}_{std}|_{(0,1)}$

9. inf union of cpt set may not be cpt

Eg: $X = \mathbb{R}$, $K_i = [i, i+1]$

Ex 8

2. Yes.

3. b) $x_{2n} = 2$, $x_{2n+1} = \frac{1}{2}$, c). No.

4. b) May not be homeomorphic.

Eg: $X = [0,1]$ $Y = [0,1)$, "$\sim$" on X is identifying 0 & 1

5. $X \cong \mathbb{Z} \not\cong Y$

6. No inclusion relations between $A^\circ/\sim$ & $(A/\sim)^\circ$.

Eg 1: $X = \mathbb{R}$, $x \sim y$ iff $x - y \in \mathbb{Z}$.

$A = [0,1)$

Eg 2: $X = [0,1]$, $x \sim y$ iff $x=0, y=1$ or $x=1, y=0$

$A = [0, \frac{1}{2})$

Ex 7

2. $\mathrm{Int}_A(B) \supset \mathrm{Int}(B)$, $\mathrm{Cl}_A(B) \subset \mathrm{Cl}(B)$

To see $\mathrm{Int}_A(B) \underset{\uparrow}{\neq} \mathrm{Int}(B)$ & $\mathrm{Cl}_A(B) \neq \mathrm{Cl}(B)$

Not necessarily

consider $(X = \mathbb{R}, \mathcal{T}_{std})$
$A = [0,1]$
$B = [0, 0.5)$

& $(X = \mathbb{R}, \mathcal{T}_{std})$
$A = (0,1)$
$B = (0, 0.5]$

3. it is the smallest topo on A s.t inclusion is continuous

6. Yes

8. $\mathcal{T}^*$ contains the product topo.

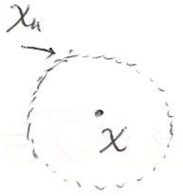
Ex 6

3. Not necessarily, depending on how to choose product metric.
Hint: consider $X = (-\frac{\pi}{2}, \frac{\pi}{2})$, metric d_{std} or d_{tan}
$\uparrow$
8. No. $d_{tan}(x,y) = |\tan x - \tan y|$

15. If either X or Y is of 1st Cat, so is $X \times Y$

18. a). No, (const map)
b). No., $f: \mathbb{R} \to \{0\}$
c). may be dense or not. & nowhere dense or not
Eg: $f = \mathrm{Id}$. Eg: $h: \mathbb{N} \to \mathbb{R}$.
$x \mapsto x$
d). may be of 1st or 2nd Cat.
Eg: $f = \mathrm{Id}$. Eg: $f = \text{const}$.

Ex 5.

1. No. Eg: 

7. See answers in Test 1, 2.

8. No. see Tutorial.

Ex 4.

2. Yes. Let $X = Y$, $f = Id$ & $d_X = d_Y = d_{discrete}$

$$d_{dis}(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

4. Yes using $\begin{cases} f^{-1}(S_1 \cap S_2) = f^{-1}(S_1) \cap f^{-1}(S_2) \\ f^{-1}(\bigcup S_\alpha) = \bigcup f^{-1}(S_\alpha) \end{cases}$

13. No, by continuity of f.

17. Yes. if target in $\mathbb{R}^n$.
No if target is S^n.

18. $f(x,y) = \dfrac{1}{x^2+y^2}$

19. $f = Id: (X, \mathcal{T}_{discrete}) \to (X, \mathcal{T}_{trivial})$, $\mathcal{T}_{trivial} = \{\phi, X\}$

(Let X contains at least two elements)

Ex 3

1. By considering (closed pt) $\rightsquigarrow$ $\mathcal{T} = \mathcal{T}_{discrete}$.

2. $\mathcal{T} = \{X, \phi\}$.

3. Eg: $X = \mathbb{R}$. B_1 = set of open intervals
$B_2 = \{ [a,b) \mid a < b \in \mathbb{R} \}$

$B_1 \cap B_2 = \phi$.

4. Yes.

13. c) Not nece separable.

Ex 2.

5. a), b), c), e) are true

for d). $(A \cap B)^\circ \overset{=}{\neq} A^\circ \cap B^\circ$, & $\overline{A \cup B} = \bar{A} \cup \bar{B}$

7. $(X = \mathbb{R}, d_{std})$ a) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ b) $\{n\}_{n \in \mathbb{N}}$.

8. consider discrete metric $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$

9. a). No such e.g. b). Not true. ($U = \mathbb{R}^2 \setminus \{(0,0)\}$, $\bar{U} = \mathbb{R}^2$)

c). Not true ($A = B = \mathbb{Q}$) in $\mathbb{R}_{std}$.

10. Eg 1: $A = \mathbb{Q}$ in $\mathbb{R}^1$ $(\bar{A})^\circ = \mathbb{R}$. $\overline{(A^\circ)} = \phi$.

Eg 2: $A = [0,1]$ in $\mathbb{R}^1$ $(\bar{A})^\circ = (0,1)$. $\overline{(A^\circ)} = [0,1]$